Lifting a 5-dimensional representation of M_{11} to a complex unitary representation of a certain amalgam

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Abstract

We lift the 5-dimensional representation of M_{11} in characteristic 3 to a unitary complex representation of the amalgam $\mathrm{GL}(2,3) *_{D_8} S_4$.

1 The representation

It is well known that the Mathieu group M_{11}, the smallest sporadic simple group, has a 5-dimensional (absolutely) irreducible representation over GF(3) (in fact, there are two mutually dual such representations). It is clear that this does not lift to a complex representation, as M_{11} has no faithful complex character of degree less than 10.

However, M_{11} is a homomorphic image of the amalgam $G = \mathrm{GL}(2,3) *_{D_8} S_4$, and it turns out that if we consider the 5-dimension representation of M_{11} as a representation of G, then we may lift that representation of G to a complex representation. We aim to do that in such a way that the lifted representation is unitary, and we realise it over $\mathbb{Z}[\frac{1}{\sqrt{-2}}]$, so that the complex representation admits reduction (mod p) for each odd prime. These requirements are stringent enough to allow us explicitly exhibit representing matrices. It turns out that reduction (mod p) for any odd prime p other than 3 yields either a 5-dimensional special linear group or a 5-dimensional special unitary group, so it is only the behaviour at the prime 3 which is exceptional.

We are unsure at present whether the 5-dimensional complex representation of G is faithful (though it does have free kernel), so we will denote the image of G in $\mathrm{SU}(5, \mathbb{Z}[\frac{1}{\sqrt{-2}}])$ by L, and denote the image of L under reduction (mod p) by L_p.

We recall that to construct a 5-dimensional representation of G, we need to construct 5-dimensional representations of $H = \mathrm{GL}(2,3)$ and $K = S_4$ which agree on a common dihedral subgroup of order 8.

We recall that H has a presentation:

$$\langle b, c : b^2 = c^3 = (bc)^8 = [b, (bc)^4] = [c, (bc)^4] = 1 \rangle,$$

for this is a presentation of a double cover of S_4 in which the pre-image of a transposition has order 2. It is also helpful in what follows to note that a unitary 2×2 matrix of trace $\pm\sqrt{-2}$ and determinant -1 has order 8 and that a unitary 2×2 matrix of trace -1 and determinant 1 has order 3. We set

$$a = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \ b = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$c = \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{-1}{\sqrt{-2}} & 0 & 0 \\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{\sqrt{-2}} & 0 & 0 \\ \frac{1}{\sqrt{-2}} & \frac{1}{\sqrt{-2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1-\sqrt{-2}}{2} & \frac{-1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{-1+\sqrt{-2}}{2} \end{pmatrix}, \ d = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{-1}{2} & \frac{-1-\sqrt{-2}}{2} & 0 & 0 \\ 0 & \frac{1-\sqrt{-2}}{2} & \frac{-1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We note that a has order 4, that b has order 2, and that c and d each have order 3. Also, bc has order 8, and $(bc)^4$ commutes with both b and c. Hence $H = \langle b, c \rangle \cong \mathrm{GL}(2,3)$.

It is clear that $K = \langle a, b, d \rangle \cong S_4$, since ad has order 2. Also, $a = (c^{-1}bc^{-1})^2$. Hence $H \cap K \geq \langle a, b \rangle$. But $K \not\subseteq H$, since there are H-invariant subspaces which are not K-invariant. Hence $H \cap K = \langle a, b \rangle$ is dihedral of order 8, so $L = \langle H, K \rangle$ is a homomorphic image of G via this representation. Furthermore, the kernel of the homomorphism is free as $\mathrm{GL}(2,3)$ and S_4 are faithfully represented. Note that, although the generator a is redundant, (as is the generator b), the presence of a and b makes it clear that L is a homomorphic image of the amalgam G.

2 Reductions (mod p)

We now discuss the groups L_p, where p is an odd prime. More precisely, we reduce the given representation (mod π), where π is a prime ideal of $\mathbb{Z}[\sqrt{-2}]$ containing the odd rational prime p. It is clear that L_3 is a subgroup of $\mathrm{SL}(5,3)$ (and choosing different prime ideals containing 3 leads to representations dual to each other). Computer calculations with GAP confirm that $L_3 \cong M_{11}$. (I am indebted to M. Geck for assistance with this computation). Suppose from now on that $p > 3$. If $p \equiv 1$ or 3, (mod 8), then -2 is a square in $\mathrm{GF}(p)$. If $p \equiv 5$ or 7, (mod 8), then -2 is a non-square in $\mathrm{GF}(p)$. Hence L_p is a subgroup of $\mathrm{SL}(5,p)$ when $p \equiv 1$ or 3 (mod 8) and L_p is a subgroup of $\mathrm{SU}(5,p)$ when $p \equiv 5$ or 7 (mod 8). We will prove:

Theorem 1

i) $L_3 \cong M_{11}$
ii)$L_p \cong \mathrm{SL}(5,p)$ *when* $p > 3$ *and* $p \equiv 1$ *or* 3 *(mod* 8*).*
iii) $L_p \cong \mathrm{SU}(5,p)$ *when* $p \equiv 5$ *or* 7 *(mod* 8*).*

Remarks: We note, in particular, that the Theorem implies that L is infinite, although we need to establish this fact during the proof in any case. We also note that G is not isomorphic to $\mathrm{SU}(5,\mathbb{Z}[\frac{1}{\sqrt{-2}}])$, since G contains no elementary Abelian subgroup of order 8 (since it is an amalgam of finite groups, neither of which contains such a subgroup), but $\mathrm{SU}(5,\mathbb{Z}[\frac{1}{\sqrt{-2}}])$ contains elementary Abelian subgroups of order 16. In fact, the theorem also implies that L is not isomorphic to $\mathrm{SU}(5,\mathbb{Z}[\frac{1}{\sqrt{-2}}])$, since all elementary Abelian 2-subgroups of L map isomorphically into L_3, and L_3 contains no elementary Abelian subgroup of order 8. We recall, however, that, as noted in [5], J-P. Serre has proved that G is isomorphic to $\mathrm{SU}(3,\mathbb{Z}[\frac{1}{\sqrt{-2}}])$.

We note also that G has the property that all of its proper normal subgroups are free. Otherwise, there is such a normal subgroup N that contains an element of order 2 or an element of order 3. All involutions in G are conjugate, because G has a semi-dihedral Sylow 2-subgroup with maximal fusion system. Both S_4 and $\mathrm{GL}(2,3)$ are generated by involutions so if N contains an involution, we obtain $N = G$. Now G has two conjugacy classes of subgroups of order 3, so if N contains an element of order 3, then N contains a subgroup isomorphic to A_4 or to $\mathrm{SL}(2,3)$, so contains an involution, and $N = G$ in that case too.

Now we proceed to prove that L is infinite. It is clear that L is irreducible, and primitive, as a linear group. We will prove more generally that no finite homomorphic image of G has a faithful complex irreducible representation of degree 5. If M were such a homomorphic image then we would have $M = [M,M]$ and M is primitive as a linear group (otherwise M would have a homomorphic image isomorphic to a transitive subgroup of S_5, which must be isomorphic to $A_{5,}$ as M is perfect). But $M \cong G/N$ for some free normal subgroup N of G, so that M has subgroups isomorphic to S_4 and $\mathrm{GL}(2,3)$, a contradiction.

Now R. Brauer (in [2]), has classified the finite primitive subgroups of $\mathrm{GL}(5,\mathbb{C})$, so we make use of his results. If $O_5(M) \not\subseteq Z(M)$, then $M/O_5(M)$, being perfect, must be isomorphic to $\mathrm{SL}(2,5)$, since $O_5(M)$ is irreducible, and has a critical subgroup of class 2 and exponent 5 on which elements of M of order prime to 5 act non-trivially. But $M/O_5(M)$ contains an isomorphic copy of $\mathrm{GL}(2,3)$, a contradiction, as $\mathrm{SL}(2,5)$ has no element of order 8.

Hence M must be isomorphic to one of $A_6, \mathrm{PSU}(4,2)$ or $\mathrm{PSL}(2,11)$. We have made use of the fact that the 5-dimensional irreducible representation of A_5 is imprimitive. We also use transfer to conclude that $Z(M)$ is trivial. Since $M = [M,M]$, we see that the given representation is unimodular, so $Z(M)$ has order dividing 5. But since $M/Z(M)$ has a Sylow 5-subgroup of order 5, when S is a Sylow 5-subgroup of G, we have $Z(M) \cap S = M' \cap Z(M) \cap S \leq S' = 1$,

as S is Abelian. Now none of $A_6, \mathrm{PSU}(4,2)$ or $\mathrm{PSL}(2,11)$ contain an element of order 8, whereas M contains a subgroup isomorphic to $\mathrm{GL}(2,3)$, and does contain an element of order 8. Hence M must be infinite, as claimed (we note that Brauer's list contains $O_5(3)'$, but this is isomorphic to $\mathrm{PSU}(4,2)$, which we have dealt with, and the realization as $\mathrm{PSU}(4,2)$ makes it clear that it can contain no element of order 8).

Now we proceed to prove that L_p is as claimed for primes $p > 3$. We note that L_p has order divisible by p since otherwise L_p is isomorphic to a finite subgroup of $\mathrm{GL}(5,\mathbb{C})$, which we have excluded above, as L_p is a homomorphic image of G. Now L_p is clearly absolutely irreducible as a linear group in characteristic p, and L_p is also primitive as a linear group, since we have already noted that no homomorphic image of G is isomorphic to a transitive subgroup of S_5. Let F_p denote the Fitting subgroup of L_p. If F_p is not central in L_p, then F_p must be a non-Abelian 5-group, and we see that L_p/F_p is isomorphic to $\mathrm{SL}(2,5)$, a contradiction, as before. Thus L_p has a component $E_p = E$, which still acts absolutely irreducibly by Clifford's Theorem. Hence the component E is unique. Since L_p is perfect, and L_p/E is solvable (using the Schreier conjecture), we see that $E = L_p$, and that L_p is quasi-simple. It is clear that L_p is a subgroup of $\mathrm{SL}(5,p)$ if $p \equiv 1,3 \pmod 8$, and a subgroup of $\mathrm{SU}(5,p)$ if $p \equiv 5,7 \pmod 8$.

By a slight abuse, we still let a,b,c,d denote their images in E, for ease of notation. We note that $X = C_E(a^2)$ is still completely reducible, since it acts irreducibly on each eigenspace of a^2. Hence $O_p(X) = 1$. Suppose that X contains an element y of order p. Then since $p \geq 5$, y must centralize $F(X)$ by the Hall-Higman Theorem. Since $O_p(X) = 1$, X must have a component, T, say. If T has a unique involution, say t, then t acts trivially on the 1-eigenspace of a^2 by unimodularity, so t must act as multiplication by -1 on the -1 eigenspace of a^2, and in fact $t = a^2$. Furthermore, T must act faithfully on the -1-eigenspace of a^2, so that $T \cong \mathrm{SL}(2,p)$ in that case.

Suppose that L_p contains no elementary Abelian subgroup of order 8. Then results of Alperin, Brauer and Gorenstein ([1]) show that L_p is isomorphic to an odd central extension of $M_{11}, \mathrm{PSU}(3,q)$, or $\mathrm{PSL}(3,q)$ for some odd q. We have excluded groups with a Sylow 2-subgroup isomorphic to a Sylow 2-subgroup of $\mathrm{PSU}(3,4)$ since L_p contains elements of order 8. Also, we know that L_p contains a semi-dihedral subgroup of order 16, so L_p does not have a dihedral Sylow 2-subgroup. Note also that L_p has centre of order dividing 5 by unimodularity. We note that since L_p contains elements of order p, we can only have $L_p \cong M_{11}$ if $p = 5$ or 11 (and in that case, L_p has trivial centre by a transfer argument). In fact, using [3], for example, M_{11} has no faithful 5-dimensional representation in any characteristic other than 3, so we can exclude that possibility. Likewise, we do not need to concern ourselves with $\mathrm{PSL}(3,3)$ or $\mathrm{PSU}(3,3)$, using the Modular Atlas ([3]). In the other cases, every involution of $\tilde{L}_p = L_p/Z(L_p)$ has a component $\mathrm{SL}(2,q)$ (note that $\tilde{L}_p$ has a single conjugacy class of involutions). In fact, it follows from inspection of the given representation that every involution of L_p has a component isomorphic to $\mathrm{SL}(2,q)$, since a central element of order

5 does not have unimodular action on any eigenspace of an involution. Now let $q = r^m$ for some odd prime r. If $r \neq p$, then $\mathrm{SL}(2, r)$ has a 2-dimensional complex representation so $r \leq 5$. However, we can exclude $r \leq 5$ using [3]. This leaves $r = p$, and $\tilde{L}_p \cong \mathrm{PSL}(3, p)$ or $\mathrm{PSU}(3, p)$. However, for $p > 5$, as noted by R. Steinberg, the Schur multiplier of $\mathrm{PSL}(3, p)$ or $\mathrm{PSU}(3, p)$ has order dividing 3, and (using [4], for example), the only non-trivial irreducible modules of dimension less than 6 for either of these groups are the natural module and its dual (note that the dual is also the Frobenius twist in the unitary case).

Suppose then that L_p contains an elementary Abelian subgroup of order 8. Then L_p contains an involution t which has the eigenvalue -1 with multiplicity 4 and the eigenvalue 1 with multiplicity 1 (the Brauer character can't take the value 1 on every non-identity element of an elementary Abelian subgroup of order 8). Then $L_p \times \langle -I \rangle$ is generated by its reflections.

By the results of Zalesskii and Serezhkin [6], we may conclude that $L_p \cong \mathrm{SL}(5, p)$ or $\mathrm{SU}(5, p)$. Several of the options from [6] are eliminated in our situation. For example, we have already that L_p is not liftable to a finite complex linear group, and it is clear that L_p is not a covering group of an alternating group (for such an alternating group would have to be of degree at most 7 and contains no element of order 8). We also note that L_p is not conjugate to an orthogonal group in odd characteristic, because bc is an element of order 8 whose eigenvalues other than -1 do not occur in mutually inverse pairs. Its eigenvalues are $-1, \alpha^2, \alpha^{-2}, \alpha, \alpha^3$ for some primitive 8-th root of unity α.

3 Concluding remarks

One way to see that L_3 is isomorphic to M_{11} is to reduce the representation modulo the ideal $(1 + \sqrt{-2})$, which clearly realizes L_3 as a subgroup of $\mathrm{SL}(5, 3)$. It turns out that L_3 has one orbit of length 11 on the 1-dimensional subspaces of the space acted upon (the other orbit being of length 110), and the resulting permutation group on the 11 subspaces of that orbit is M_{11}. In reality, it is knowledge of this representation which led to the attempt to lift it to a complex representation of the amalgam.

As we remarked earlier, we are unsure at present whether the representation of G afforded by L is a faithful one. Consequently, while we know that all proper normal subgroups of G are free, we have not proved that this is the case for L. We therefore feel it is worth noting:

Theorem 2: *Neither G nor L has any non-identity solvable normal subgroup.*

Proof: This is clear for G, but for completeness we indicate a proof. Every proper normal subgroup of G is free. Hence if $1 \neq S \lhd G$, is solvable, then S is free of rank one. But $G = [G, G]$, so that $S \leq Z(G)$. Now suppose that there is a non-identity element $s \in S$, and recall that G has the form $H *_D K$, where $H \cong \mathrm{GL}(2, 3)$, $K \cong S_4$ and $D = H \cap K$ is dihedral with 8 elements. Now since s has infinite order, s may be expressed in the form $s = dx_1 x_2 \ldots x_m x_{m+1}$, where

$d \in D, m \geq 1$ and each $x_i \in (H \cup K)\backslash D$ but there is no value of i for which both x_i and x_{i+1} both lie in H, and there is no value of i for which x_i and x_{i+1} both lie in K. The expression is not unique, but for each i, the right coset of D containing x_i (in whichever of H or K contains x_i) is uniquely determined.

But for any $c \in D$, we have $s = s^c = d^c x_1^c x_2^c \ldots x_{m+1}^c$. It follows that $x_i^c x_i^{-1} \in D$ for each i and each $c \in D$. Hence each x_i normalizes D. But D is self-normalizing in K and $N_H(D)$ is semi-dihedral of order 16, so that $s \in N_H(D)$, a contradiction, as s has infinite order.

As for L, note that if $S \lhd L$ is solvable, then $[L, S]$ is in the kernel of each reduction (mod p), as L_p is always quasi-simple. However, given a matrix $x \in L$, there is a minimal non-negative integer s such that $2^s x$ has all its entries in $\mathbb{Z}[\sqrt{-2}]$. Now if $x \neq I$, then there are only finitely many prime ideals of $\mathbb{Z}[\sqrt{-2}]$ which contain all entries of $2^s x - 2^s I$. Hence $[L, S] = I$. But, as L is an irreducible linear group, $Z(L)$ consists of scalar unitary matrices of determinant 1 with entries in $\mathbb{Q}[\sqrt{-2}]$, so $Z(L) = 1$.

Remark: It might also be worth noting that Theorem 1 implies that the only torsion that L can have is 2-torsion, 3-torsion, or 5-torsion. Only elements of 3-power order can be in the kernel of reduction (mod 3), so the only possibilities for prime orders of elements of L are $2, 3, 5$ or 11. But any element of order 11 in L would have trace an irrational element of $\mathbb{Q}[\sqrt{-11}]$, while its trace must be in $\mathbb{Q}[\sqrt{-2}]$. At present, we see no obvious way to prove that L has no 5-torsion, since L_p always contains elements of order 5. We do note that L does not contain the obvious permutation matrix $f = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$, since $\langle b, f \rangle$ contains an elementary Abelian subgroup of order 16 and L does not.

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